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Extended Double Soliton Solution Families for the Statically Axisymmetric Self-Dual *SU*(2) Gauge Field Equations

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By using the extended double complex function method, the statically axisymmetric self-dual $SU(2)$ gauge field equations and, in turn, the Belinsky–Zakharov solution-generating technique are generalized to extended double forms. The restriction on "soliton index" in the original solution-generating technique is eliminated so that for each positive integer, we can obtain *physical* soliton solutions of the statically axisymmetric self-dual $SU(2)$ gauge field equations in pairs. Some sufficient conditions are given for seed solutions, with which the corresponding scattering wave functions can be written out directly. As examples, some soliton solution families are given, most solutions of which are new.

1. INTRODUCTION

The statically axisymmetric self-dual $SU(2)$ gauge field equations, SAS-DSU(2)GFEs, on four-Euclidean space can be written as (Yang, 1977; Letelier, 1982)

$$\begin{gathered} f\nabla^2 f - \nabla f \cdot \nabla f + \nabla g \cdot \nabla \bar{g} = 0 \\ f\nabla^2 g - 2\nabla f \cdot \nabla g = 0, \qquad f\nabla^2 \bar{g} - 2\nabla f \cdot \nabla \bar{g} = 0 \end{gathered} \tag{1.1}$$

where $f = f(\rho, z)$ is a real, and $g = g(\rho, z)$ a complex, function of the cylinder coordinates ρ and z. Here ∇ and ∇^2 denote, respectively, the gradient and Laplace operators with respect to the flat three-dimensional metric:

$$dl^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2 \tag{1.2}$$

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Letelier (1982) gave an inverse scattering method (ISM) for obtaining soliton solutions of equation (1.1) which extends the Belinsky–Zakharov (BZ) method (Belinsky and Zakharov, 1978, 1979). However, as a result of the reality requirement of physical solutions, when starting from a physical seed solution, we can obtain a physical n-soliton solution only for n an even number. If we try to obtain physical soliton solutions with odd-number indices, we must start from a nonphysical seed solution, but in this case, the soliton solutions with even-number indices are nonphysical. Zhong (1985) suggested a *double complex function method*, realized an analytic continuation of the NK substitution (Neugebauer and Kramer, 1969), and overcame similar difficulties in the theory of reduced gravitation fields (Zhong, 1988; Gao and Zhong, 1992). However, since the function $g(\rho, z)$ in equation (1.1) is complex, the double complex method in Zhong (1985) is no longer valid for the self-dual $SU(2)$ gauge fields unless we consider only the special case of $g(\rho, z) = \sigma(\rho, z)e^{i\alpha}$ with $\sigma(\rho, z)$ being a real function and α a real constant (Witten, 1979; Zhong, 1989), but we shall *not* make this restriction in the present paper. Moreover, we noted that if (f, g) is a solution of equation (1.1), so is $(f, e^{i\theta}g)$ with θ a real constant. Therefore, in common schemes there is even no homologue of the NK substitution for equation (1.1). Recently, we suggested an extended double complex function method and used it to obtain new symmetries of the reduced gravitation field equations (Gao *et al.*, 1997). In the present paper, we shall use this method to study the SASDSU(2)GFEs and find that this enable us to overcome all of the above-mentioned difficulties and obtain some new results.

In the Preliminaries below, we recall some relevant concepts and results. In Section 2, we give an extended double form of the SASDSU(2)GFEs, where the extended NK substitution is found and its "extended analytic continuation" is realized automatically. In Section 3, the BZ ISM is generalized to an extended double form, which enables us to get rid of the restriction on the soliton index. Starting from an extended double-seed solution, we can obtain a pair of physical soliton solutions of the SASDSU(2)GFEs for every positive integer n. The "scattering wave function" is the keystone of the ISM, but obtaining it in general is very difficult. In Section 4, we study the Lax pair of the extended double ISM and find that for some kinds of seed solutions the scattering wave functions can be obtained directly (one need not solve concretely the scattering equations). Finally, in Section 5 we give as examples some soliton solution families, most of which are new.

Preliminaries

Here we briefly recall relevant concepts and notations of the extended double (ED) complex function method from Gao *et al.*, (1997). Essentially, the ED complex number ring (denoted by EDC) is a continuation of the original double complex number ring given in Zhong (1985). More explicitly, let i and J denote, respectively, the ordinary and the ED imaginary unit, i.e., $J = j$ ($j^2 = -1, j \neq \pm i$) or $J = \epsilon$ ($\epsilon^2 = +1, \epsilon \neq \pm 1$). If a series $\Sigma_{n=0}^{\infty}|a_n|$, $a_n \in \mathbf{C}$ (ordinary complex number field) is convergent, then $a(J) = \Sigma_{n=0}^{\infty} a_n J^{2n}$ is called a double ordinary complex (DOC) number, which corresponds to a pair (a_C, a_H) of ordinary complex numbers, where $a_C := a(J = j)$, $a_H := a(J = \epsilon)$. When $a(J)$ and $b(J)$ both are DOC numbers, then

$$c(J) = a(J) + Jb(J) \tag{1.3}$$

is called an ED complex number; it corresponds to a pair (c_C, c_H), where $c_C := c(J = j) = a_C + jb_C$ and $c_H := c(J = \epsilon) = a_H + \epsilon b_H$ are, respectively, an extended elliptic complex number and an extended hyperbolic complex number. We say that c_C, c_H are dual to each other. All ED-complex numbers with usual addition and multiplication constitute a commutative ring, namely, the above-mentioned ring EDC.

The ring EDC involves two imaginary units i and J, so we have two complex conjugation operations "—" and "~" which act on i and J, respectively, i.e.,

$$\overline{c(J)} := \overline{a(J)} + J\overline{b(J)}, \qquad \tilde{c(J)} := a(J) - Jb(J) \tag{1.4}$$

where $\overline{a(J)}$ and $\overline{b(J)}$ are ordinary complex conjugations of $a(J)$ and $b(J)$. Note that, by definition, the operations "—" and "~" have nothing to do with each other.

If in equation (1.3), $a(J)$ and $b(J)$ are DOC functions of some ordinary complex (or real) variables $z_1, \ldots, z_n$, then $c(z_1, \ldots, z_n; J) = a(z_1, \ldots, z_n; J) + Jb(z_1, \ldots, z_n; J)$ is called an ED-complex function and we say $c(z_1, \ldots, z_n; J)$ is continuous, analytical, etc., iff $a(z_1, \ldots, z_n; J)$ and $b(z_1, \ldots, z_n; J)$ both, as ordinary complex functions, have the same properties. If $a(J)$, $b(J)$, and $z_1, \ldots, z_n$ are all restricted in $\mathbf{R}$ (real number field), then the theory above is reduced to the one introduced in Zhong (1985).

In addition, in this paper we shall also use the commutation operation "∘" of the ED imaginary units j and ϵ, which is defined as

$$\circ\colon \; J \to \mathring{J}; \qquad \mathring{j} = \epsilon, \qquad \mathring{\epsilon} = j \tag{1.5}$$

2. EXTENDED DOUBLE FORM OF THE SASDSU(2)GFES

In order to obtain an extended double form of the SASDSU(2)GFEs, we consider the following set of equations:

$$\begin{gathered} f'\nabla^2 f' - \nabla f' \cdot \nabla f' - \nabla g' \cdot \nabla \overline{g'} = 0 \\ f'\nabla^2 g' - 2\nabla f' \cdot \nabla g' = 0, \qquad f'\nabla^2 \overline{g'} - 2\nabla f' \cdot \nabla \overline{g'} = 0 \end{gathered} \tag{2.1}$$

where $f' = f'(\rho, z)$ and $g'(\rho, z)$ are, respectively, real and ordinary complex functions of the real coordinates ρ and z. Equation (2.1) implies that there exists a function $G = G(\rho, z)$ satisfying

$$\partial_\rho G = \rho f'^{-2}\partial_z g', \qquad \partial_x G = -\rho f'_{-2}\partial_\rho g' \tag{2.2a}$$

In other words, we can introduce a transformation V of f', g' defined as

$$V\colon \; f', g' \to G = V(f', g') := \int \rho f'^{-2}(\partial_z g' \cdot d\rho - \partial_\rho g' \, dz) \tag{2.2b}$$

such that equation (2.1) is invariant under the mapping

$$f' \to F := \rho f'^{-1}, \qquad g' \to jG \tag{2.3}$$

This is a generalization of the NK substitution (Neugebauer and Kramer, 1969) to an extended elliptic complex form. Now we extend "analytically" the function pair (F, jG) to an ED-complex function pair, i.e.,

$$(F, jG) \to (F(J), JG(J)) \tag{2.4}$$

where $F(J) = F(\rho, z; J)$ and $G(J) = G(\rho, z; J)$ are, respectively, extended double real and DOC functions of the real coordinates ρ and z.

Owing to equations (2.3) and (2.4), equation (2.1) is immediately extended to an ED form as

$$\begin{aligned} F(J)\nabla^2 F(J) - \nabla F(J) \cdot \nabla F(J) - J^2 \nabla G(J) \cdot \nabla \overline{G(J)} &= 0 \\ F(J)\nabla^2 G(J) - 2\nabla F(J) \cdot \nabla G(J) &= 0 \\ F(J)\nabla^2 \overline{G(J)} - 2\nabla F(J) \cdot \nabla \overline{G(J)} &= 0 \end{aligned} \tag{2.5}$$

When $J = j$ and $J = \epsilon$, equation (2.5) gives, respectively, equations (1.1) and (2.1). That is, equations (1.1) and (2.1) are "analytically" linked by J and are dual to each other.

From the discussions above and due to equations (2.2) and (2.3), we obtain the following theorem.

Theorem 1. If an ED solution $(F(J), G(J))$ of equation (2.5) is known, then a pair of solutions of SASDSU(2)GFEs can be obtained as

$$\begin{cases} f(\rho, z) = F_C(\rho, z) = F(\rho, z; J = j) \\ g(\rho, z) = G_C(\rho, z) = G(\rho, z; J = j) \end{cases} \tag{2.6a}$$

$$\begin{cases} \hat{f}(\rho, z) = \rho/F_H(\rho, z) = \rho/F(\rho, z; J = \epsilon) \\ \hat{g}(\rho, z) = V(F_H, G_H) \end{cases} \tag{2.6b}$$

where the transformation V is defined in equation (2.2).

Thus from a single ED solution of equation (2.5) we can obtain two distinct solutions of the SASDSU(2)GFEs. Furthermore, these two solutions both corresponds to *physical* fields. Therefore we have realized automatically an "analytic" continuation of the extended NK substitution (2.3).

3. ED SOLITON SOLUTION-GENERATING ALGORITHM

It is usually not easy to solve equation (2.5). In this section, we shall generalize the BZ ISM into an ED form and discuss its effect.

We introduce a double ordinary Hermitian 2×2 matrix function

$$M(J) = M(\rho, z; J) = \frac{1}{F(J)} \begin{pmatrix} 1 & G(J) \\ \overline{G(J)} & G(J)\overline{G(J)} - J^2F^2(J) \end{pmatrix} \tag{3.1}$$

Then equation (2.5) can be written in an ED BZ form as

$$\partial_\rho[\rho\partial_\rho M(J) \cdot M^{-1}(J)] + \partial_z[\rho\partial_z M(J) \cdot M^{-1}(J)] = 0 \tag{3.2a}$$

$$M^\dagger(J) = M(J) \tag{3.2b}$$

$$\det M(J) = -J^2 \tag{3.2c}$$

where "$\dagger$" denotes the ordinary Hermitian conjugation. Conversely, if $M(\rho, z; J)$ is an ED solution of equations (3.2), then

$$F(J) = \frac{1}{[M(J)]_{11}}, \qquad G(J) = \frac{[M(J)]_{12}}{[M(J)]_{11}} \tag{3.3}$$

satisfies equation (2.5).

Equation (3.2a) is the integrability condition of the following ED Lax pair:

$$D_\rho\Psi(\lambda; J) = \frac{\rho U(J) + \lambda W(J)}{\rho^2 + \lambda^2}\Psi(\lambda; J) \tag{3.4}$$

$$D_z\Psi(\lambda; J) = \frac{\rho W(J) - \lambda U(J)}{\rho^2 + \lambda^2}\Psi(\lambda; J)$$

where

$$D_\rho := \partial_\rho + \frac{2\lambda\rho}{\rho^2 + \lambda^2}\partial_\lambda, \qquad D_z := \partial_z - \frac{2\lambda^2}{\rho^2 + \lambda^2}\partial_\lambda \tag{3.5}$$

$$U(J) := \rho\partial_\rho M(J) \cdot M^{-1}(J), \qquad W(J) := \rho\partial_z M(J) \cdot M^{-1}(J)$$

and $\Psi(\lambda; J) = \Psi(\rho, z, \lambda; J)$ is an ED-complex 2×2 matrix function of ρ, z, and an ordinary complex spectral parameter λ. Since by definition (3.5),

$\widetilde{D}_\rho = D_\rho$, $\widetilde{D}_z = D_z$, $\widetilde{U(J)} = U(J)$, and $\widetilde{W(J)} = W(J)$, without loss of generality, in the following we compatibly select $\Psi(\lambda; J)$ satisfying $\widetilde{\Psi(\lambda; J)} = \Psi(\lambda; J)$, i.e., being a DOC matrix function. By the treatments similar to that of Belinsky and Zakharov (1978, 1979) and Letelier (1982), we find that if $\Psi_0(\lambda; J)$ is a solution of equation (3.4) for a known $M(J)$, say $M_0(J)$, then the ED n-soliton solution $M_n(J)$ of equations (3.2a), (3.2b) can be obtained as follows:

$$\begin{aligned}
M_n(J) &= |\det M'_n(J)|^{-1/2} \cdot M'_n(J) \\
[M'_n(J)]_{ab} &= [M_0(J)]_{ab} - \sum_{k,l=1}^{n} \frac{\overline{N_a^{(l)}(J)}[\Gamma^{-1}(J)]_{lk} N_b^{(k)}(J)}{\mu_k \overline{\mu_l}} \\
\Gamma_{lk}(J) &= \frac{m_a^{(l)}(J)[M_0(J)]_{ab}\overline{m_b^{(k)}(J)}}{\mu_l \overline{\mu_k} + \rho^2} = \overline{\Gamma_{kl}(J)} \qquad (3.6)\\
N_a^{(k)}(J) &= m_b^{(k)}(J)[M_0(J)]_{ba}, \qquad m_a^{(k)}(J) = m_{0b}^{(k)}(J)[Q^{(k)}(J)]_{ba} \\
Q^{(k)}(J) &= \Psi_0^{-1}(\rho, z, \lambda = \mu_k; J) \\
\det M'_n(J) &= (-1)^n \rho^{2n} \prod_{k=1}^{n} |\mu_k|^{-2} \det M_0(J)
\end{aligned}$$

where the $m_{0a}^{(k)}$ are some DOC constants, the sum convention on the indices a and b is assumed (a, b = 1, 2), and

$$\begin{aligned}
\partial_\rho \mu_k &= \frac{2\rho\mu_k}{\rho^2 + \mu_k^2}, \qquad \partial_z \mu_k = \frac{-2\mu_k^2}{\rho^2 + \mu_k^2} \qquad (3.7)\\
\mu_k &= \mu_k(\rho, z) = \alpha_k - z \pm [(\alpha_k - z)^2 + \rho^2]^{1/2}
\end{aligned}$$

where the α_k are some ordinary complex constants. Considering equations (3.2c) and (3.6), we have

$$\begin{aligned}
\det[M_n(J)] &= (-1)^n \det M_0(J) \\
&= (-1)^n(-J^2) \qquad (3.8)
\end{aligned}$$

From the definition (1.5), it is obvious that if $M_n(J)$ is an ED solution of equations (3.2a) and (3.2b), so is $M_n(\mathring{J})$, and for any nonnegative integer k, we have

$$\det M_{2k}(J) = -J^2, \qquad \det M_{2k+1}(\mathring{J}) = \mathring{J}^2 = -J^2 \qquad (3.9)$$

Let

$$\mathcal{M}_n(J) = \begin{cases} M_n(J) & \text{when } n \text{ is even} \\ M_n(\mathring{J}) & \text{when } n \text{ is odd} \end{cases} \qquad (3.10)$$

Then

$$\det \mathcal{M}_n(J) = -J^2 \tag{3.11}$$

for any nonnegative integer n. Therefore we come to the conclusion that $\mathcal{M}_n(J)$ not only is an ED n-soliton solution of equations (3.2a) and (3.2b), but also always satisfies condition (3.2c) for any nonnegative integer n. Thus the BZ ISM has been extended to an ED form, and by means of the ED ISM, we have eliminated the restriction on the soliton index n in the original scheme. Once an ED n-soliton solution $\mathcal{M}_n(J)$ of equations (3.2) is obtained for a nonnegative integer n, then a pair of dual n-soliton solutions of SASD-SU(2)GFEs can be easily given by using equations (3.3) and (2.6).

4. SEED SOLUTION $M_0(J)$ AND WAVE FUNCTION $\Psi_0(\lambda; J)$

From the above discussions we can see that the key step of the ISM is to find a suitable scattering wave function $\Psi_0(\lambda; J) = \Psi_0(\rho, z, \lambda; J)$ satisfying the following system of differential equations for some seed solution $M_0(J)$:

$$D_\rho \Psi_0(\lambda; J) = \frac{\rho U_0(J) + \lambda W_0(J)}{\rho^2 + \lambda^2} \Psi_0(\lambda; J) \tag{4.1a}$$

$$D_z \Psi_0(\lambda; J) = \frac{\rho W_0(J) - \lambda U_0(J)}{\rho^2 + \lambda^2} \Psi_0(\lambda; J)$$

and initial condition

$$\Psi_0(\rho, z, \lambda = 0; J) = M_0(\rho, z; J) \tag{4.1b}$$

where

$$U_0(J) = \rho\partial_\rho M_0(J) \cdot M_0^{-1}(J), \qquad W_0(J) = \rho\partial_z M_0(J) \cdot M_0^{-1}(J) \tag{4.2}$$

However, it usually is very difficult to solve equations (4.1). Particularly for the case of nondiagonal $M_0(J) = M_0(\rho, z; J)$ the problem becomes even more complicated. Hence we wish to find some simpler methods.

Since in equations (3.6) $\Psi_0(\lambda; J)$ only enters evaluated along the pole's trajectories $\mu_k (k = 1, 2, \ldots)$, in order to compute the soliton solutions we only need $\Psi_{0k}(J) := \Psi_{0k}(\rho, z, \lambda = \mu_k; J)$ $(k = 1, 2, \ldots)$. From equations (4.1a) and (3.7), $\Psi_{0k}(J)$ satisfies

$$\partial_\rho \Psi_{0k}(J) = \frac{1}{2\mu_k} [\partial_\rho \mu_k \cdot U_0(J) - \partial_z \mu_k \cdot W_0(J)] \Psi_{0k}(J)$$

$$\partial_z \Psi_{0k}(J) = \frac{1}{2\mu_k} [\partial_\rho \mu_k \cdot W_0(J) + \partial_z \mu_k \cdot U_0(J)] \Psi_{0k}(J) \tag{4.3a}$$

The condition (4.1b) now reads

$$\Psi_{0k}(J)|_{\mu_k\to 0} = M_0(J) \tag{4.3b}$$

In general, it is still very difficult to solve equations (4.3). In the following, we shall show that for some kinds of seed solutions, $\Psi_{0k}(J)$ can be obtained directly.

Equations (3.7) give

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)\ln \mu_k(\rho, z) = 0 \tag{4.4}$$

and

$$\left.\frac{\partial_\rho \mu_k}{2\mu_k}\right|_{\mu_k\to 0} = \frac{1}{\rho}, \qquad \left.\frac{\partial_z \mu_k}{2\mu_k}\right|_{\mu_k\to 0} = 0 \tag{4.5}$$

Hence we have the following lemma.

Lemma. Let $\varphi = \varphi(\rho, z)$ be a harmonic function, i.e.,

$$\nabla^2\varphi = \left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)\varphi = 0 \tag{4.6}$$

Then equation (4.4) guarantees that the function

$$Y_k[\varphi, \mu_k] = \int \frac{\rho}{2\mu_k}[(\partial_\rho\mu_k\partial_\rho\varphi - \partial_z\mu_k\partial_z\varphi)\,d\rho + (\partial_z\mu_k\partial_\rho\varphi + \partial_\rho\mu_k\partial_z\varphi)\,dz] \tag{4.7}$$

exists, and equation (4.5) gives

$$Y_k[\varphi, \mu_k]|_{\mu_k\to 0} = \varphi \tag{4.8}$$

Further, we have the following theorems.

Theorem 2. If the seed solution $M_0(\rho, z; J)$ is dependent on ρ and z only through a harmonic function $\varphi(\rho, z)$, i.e.,

$$M_0(\rho, z; J) = M_0(\varphi; J) \tag{4.9}$$

then the corresponding scattering wave function $\Psi_{0k}(J)$ can be directly obtained as

$$\Psi_{0k}(J) = M_0(\varphi \to Y_k[\varphi, \mu_k]; J) \tag{4.10}$$

namely, the φ in $M_0(\varphi; J)$ is replaced by the function $Y_k[\varphi, \mu_k]$ defined by equation (4.7).

More generally, we have the following result.

Theorem 3. If the seed solution $M_0(\rho, z; J)$ is dependent on ρ and z through a set of harmonic functions $\varphi_1(\rho, z), \ldots, \varphi_s(\rho, z)(s \geq 1)$, i.e., $M_0(\rho, z; J) = M_0(\varphi_1, \ldots, \varphi_s; J)$, and the following condition is satisfied:

$$\frac{\partial}{\partial \varphi_j}\left[\left(\frac{\partial}{\partial \varphi_i} M_0(\varphi_1, \ldots, \varphi_s; J)\right) \cdot M_0^{-1}(\varphi_1, \ldots, \varphi_s; J)\right] = 0, \qquad i, j = 1, \ldots, s \tag{4.11}$$

then the corresponding scattering wave function can be directly obtained as

$$\Psi_{0k}(J) = M_0(\varphi_1 \rightarrow Y_k[\varphi_1, \mu_k], \ldots, \varphi_s \rightarrow Y_k[\varphi_s, \mu_k]; J) \tag{4.12}$$

Proof. The condition (4.11) implies

$$\frac{\partial}{\partial \varphi_i} M_0(\varphi_1, \ldots, \varphi_s; J) = A_i(J) M_0(\varphi_1, \ldots, \varphi_s; J), \qquad i = 1, \ldots, s \tag{4.13}$$

where, for a certain seed solution $M_0(\varphi_1, \ldots, \varphi_s; J)$, $\{A_s(J)\}$ is a certain set of 2×2 DOC constant matrices. Consequently, we have

$$U_0(J) = \sum_{i=1}^{s} A_i(J)\rho\partial_\rho\varphi_i, \qquad W_0(J) = \sum_{i=1}^{s} A_i(J)\rho\partial_z\varphi_i \tag{4.14}$$

Therefore, equations (4.3a) now read

$$\begin{aligned} \partial_\rho \Psi_{0k}(J) &= \frac{\rho}{2\mu_k}\left[\sum_{i=1}^{s} (\partial_\rho\mu_k\partial_\rho\varphi_i - \partial_z\mu_k\partial_z\varphi_i)A_i(J)\right]\Psi_{0k}(J) \\ \partial_z \Psi_{0k}(J) &= \frac{\rho}{2\mu_k}\left[\sum_{i=1}^{s} (\partial_z\mu_k\partial_\rho\varphi_i + \partial_\rho\mu_k\partial_z\varphi_i)A_i(J)\right]\Psi_{0k}(J) \end{aligned} \tag{4.15}$$

From equation (4.13) and the Lemma, we can examine directly that $\Psi_{0k}(J) = M_0(\varphi_1 \rightarrow Y_k[\varphi_1, \mu_k], \ldots, \varphi_s \rightarrow Y_k[\varphi_s, \mu_k]; J)$ is a solution of equation (4.15) and satisfies the initial condition (4.3b).

In some sense, Theorem 2 can be regarded as a special case ($s = 1$) of Theorem 3. However, in the case of Theorem 2, the condition (4.11) is always satisfied automatically.

5. SOME CONCRETE ED SOLITON SOLUTION FAMILIES OF THE SASDSU(2)GFES

According to the theorems in Section 4, for some "suitable" kinds of seed solutions, the scattering wave functions can be written out directly; then the remainders for generating new soliton solutions are only some algebraic calculations. In this section, we give some concrete soliton solution families of the SASDSU(2)GFEs; most solutions we obtain here are new and some of these new solutions cannot be obtained by the original ISM.

1. Take the ED Weyl-type solution (Weyl, 1917) of equations (3.2) as a seed solution, i.e.,

$$M_0(J) = \begin{pmatrix} e^{-\varphi} & 0 \\ 0 & -J^2 e^{\varphi} \end{pmatrix}, \qquad \nabla^2 \varphi(\rho, z) = 0 \tag{5.1}$$

By Theorem 2, the corresponding scattering wave function is

$$\Psi_{0k} = \begin{pmatrix} e^{-Y_k} & 0 \\ 0 & -J^2 e^{Y_k} \end{pmatrix} \tag{5.2}$$

where $Y_k = Y_k[\varphi, \mu_k]$ is defined by equation (4.7). From equations (3.6) and writing $p_k(J) := m_{01}^{(k)}(J)$, $q_k(J) := m_{02}^{(k)}(J)$, we obtain the related matrix elements of the ED one-soliton solution $M_1(J)$:

$$\begin{aligned}
[M_1(J)]_{11} &= \frac{|\mu_1|}{\rho} e^{-\varphi} - \frac{|\mu_1|^2 + \rho^2}{\rho|\mu_1|} \\
&\quad \times \frac{|p_1(J)|^2 \exp(Y_1 + \overline{Y_1} - 2\varphi)}{|p_1(J)|^2 \exp(Y_1 + \overline{Y_1} - \varphi) - J^2 |q_1(J)|^2 \exp[-(Y_1 + \overline{Y_1} - \varphi)]} \\
[M_1(J)]_{12} &= \overline{[M_1(J)]_{21}} \\
&= -\frac{|\mu_1|^2 + \rho^2}{\rho|\mu_1|} \\
&\quad \times \frac{\overline{p_1(J)} q_1(J) \exp(\overline{Y_1} - Y_1)}{|p_1(J)|^2 \exp(Y_1 + \overline{Y_1} - \varphi) - J^2 |q_1(J)|^2 \exp[-(Y_1 + \overline{Y_1} - \varphi)]}
\end{aligned} \tag{5.3}$$

By equations (3.10), (3.3), and (2.6), we obtain a pair of dual one-soliton solutions (f_1, g_1), $(\hat{f}_1, \hat{g}_1)$ of SASDSU(2)GFEs as

$$\begin{cases} f_1(\rho, z) = 1/[M_{1H}]_{11} \\ g_1(\rho, z) = [M_{1H}]_{12}/[M_{1H}]_{11} \end{cases} \tag{5.4a}$$

$$\begin{cases} \hat{f}_1(\rho, z) = \rho[M_{1E}]_{11} \\ \hat{g}_1(\rho, z) = V(F_{1E}, G_{1E}) \end{cases} \tag{5.4b}$$

where $F_{1E} = 1/[M_{1E}]_{11}$, $G_{1E} = [M_{1E}]_{12}/[M_{1E}]_{11}$. Similarly, the related elements of the ED two-soliton solution $M_2(J)$ are

$$
\begin{aligned}
[M_2(J)]_{11} &= \frac{|\mu_1\mu_2|}{\rho^2} e^{-\varphi} \\
&\quad - \frac{|\mu_1\mu_2|}{\rho^2\Delta(J)} \left[\frac{\overline{N_1^{(1)}(J)}N_1^{(1)}(J)}{\mu_1\overline{\mu_1}} \frac{m^{(2)}(J)\overline{m^{(2)}(J)}}{\mu_2\overline{\mu_2} + \rho^2} \right. \\
&\quad + \frac{\overline{N_1^{(1)}(J)}N_1^{(2)}(J)}{\mu_1\overline{\mu_2}} \frac{m^{(1)}(J)\overline{m^{(2)}(J)}}{\overline{\mu_1}\mu_2 + \rho^2} - \frac{\overline{N_1^{(2)}(J)}N_1^{(1)}(J)}{\mu_2\overline{\mu_1}} \frac{m^{(2)}(J)\overline{m^{(1)}(J)}}{\mu_2\overline{\mu_1} + \rho^2} \\
&\quad \left. + \frac{\overline{N_1^{(2)}(J)}N_1^{(2)}(J)}{\mu_2\overline{\mu_2}} \frac{m^{(1)}(J)\overline{m^{(1)}(J)}}{\mu_1\overline{\mu_1} + \rho^2} \right]
\end{aligned} \tag{5.5a}
$$

$$
\begin{aligned}
[M_2(J)]_{12} &= \overline{[M_2(J)]_{21}} \\
&= -\frac{|\mu_1\mu_2|}{\rho^2\Delta(J)} \left[\frac{\overline{N_1^{(1)}(J)}N_2^{(1)}(J)}{\mu_2\overline{\mu_1}} \frac{m^{(2)}(J)\overline{m^{(2)}(J)}}{\mu_2\overline{\mu_2} + \rho^2} \right. \\
&\quad - \frac{\overline{N_1^{(1)}(J)}N_2^{(2)}(J)}{\mu_1\overline{\mu_2}} \frac{m^{(1)}(J)\overline{m^{(2)}(J)}}{\mu_1\overline{\mu_2} + \rho^2} - \frac{\overline{N_1^{(2)}(J)}N_2^{(1)}(J)}{\mu_2\overline{\mu_1}} \frac{m^{(2)}(J)\overline{m^{(1)}(J)}}{\mu_2\overline{\mu_1} + \rho^2} \\
&\quad \left. + \frac{\overline{N_1^{(2)}(J)}N_2^{(2)}(J)}{\mu_2\overline{\mu_2}} \frac{m^{(1)}(J)\overline{m^{(1)}(J)}}{\mu_1\overline{\mu_1} + \rho^2} \right]
\end{aligned} \tag{5.5b}
$$

where

$$
\begin{aligned}
\Delta(J) &= \frac{m^{(1)}(J)\overline{m^{(1)}(J)}}{|\mu_1|^2 + \rho^2} \frac{m^{(2)}(J)\overline{m^{(2)}(J)}}{|\mu_2|^2 + \rho^2} - \left| \frac{m^{(1)}(J)\overline{m^{(2)}(J)}}{\mu_1\overline{\mu_2} + \rho^2} \right| = \overline{\Delta(J)} \\
m^{(k)}(J)\overline{m^{(l)}(J)} &= p_k(J)\overline{p_l(J)}e^{(Y_k+\overline{Y_l}-\varphi)} - J^2 q_k(J)\overline{q_l(J)}e^{-(Y_k+\overline{Y_l}-\varphi)} \\
N_1^{(k)}(J) &= p_k(J)e^{(Y_k-\varphi)}, \qquad N_2^{(k)}(J) = q_k(J)e^{-(Y_k-\varphi)}
\end{aligned} \tag{5.6}
$$

By equations (3.10), (3.3), and (2.6), we obtain a pair of dual two-soliton solutions (f_2, g_2), $(\hat{f}_2, \hat{g}_2)$ of SASDSU(2)GFEs as

$$
\begin{cases} f_2(\rho, z) = 1/[M_{2E}]_{11} \\ g_2(\rho, z) = [M_{2E}]_{12}/[M_{2E}]_{11} \end{cases} \tag{5.7a}
$$

$$
\begin{cases} \hat{f}_2(\rho, z) = \rho[M_{2H}]_{11} \\ \hat{g}_2(\rho, z) = V(F_{2H}, G_{2H}) \end{cases} \tag{5.7b}
$$

where $F_{2H} = 1/[M_{2H}]_{11}$, $G_{2H} = [M_{2H}]_{12}/[M_{2H}]_{11}$.

By continuing the above process, for each positive integer n, we can, in principle, obtain two dual n-soliton solutions of the SASDSU(2)GFEs. As a special case, if we take $\varphi(\rho, z) = az + b \ln \rho + c(1/2\rho^2 - z^2)$, where a, b, and c are constants, then equations (5.4a) and (5.7a) give the results of Letelier (1982). However, the solutions (5.4b) and (5.7b) are new, and all solutions here need not start from a nonphysical seed solution.

2. Let the seed solution be

$$M_{0H}(\rho, z) = \begin{pmatrix} \varphi & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}, \qquad \nabla^2\varphi(\rho, z) = 0 \tag{5.8}$$

where α is a real constant and the subscript "H" of M_{0H} indicates that equation (5.8) is a solution of equations (3.2) only when $J = \epsilon$. By Theorem 2, the corresponding wave function is

$$\Psi_{0kH} = \begin{pmatrix} Y_k & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \tag{5.9}$$

where $Y_k = Y_k[\varphi, \mu_k]$ is defined by equation (4.7). From equations (3.6) (with $J = \epsilon$) and writing $p_k = m_{01H}^{(k)}$ and $q_k = m_{02H}^{(k)}$, we obtain the related matrix elements of the one-soliton solution M_{1H} as

$$[M_{1H}]_{11} = -\frac{1}{\rho|\mu_1|} \times \frac{(|\mu_1|^2 + \rho^2)|p_1 - q_1 e^{-i\alpha} Y_1|^2 + \rho^2 [|g_1|^2(\varphi - Y_1 - \overline{Y_1}) + 2\,\mathrm{Re}(p_1 \overline{g_1} e^{i\alpha})]\varphi}{|q_1|^2(\varphi - Y_1 - \overline{Y_1}) + 2\,\mathrm{Re}(p_1 \overline{q_1} e^{i\alpha})}$$

$$[M_{1H}]_{12} = \overline{[M_{1H}]_{12}} = \frac{e^{i\alpha}}{\rho|\mu_1|} \frac{|q_1|^2[\rho^2(\overline{Y_1} - \varphi) - |\mu_1|^2 Y_1] + [|\mu_1|^2 p_1 \overline{q_1} - \rho^2 q_1 \overline{p_1}] e^{i\alpha}}{|q_1|^2(\varphi - Y_1 - \overline{Y_1}) + 2\,\mathrm{Re}(p_1 \overline{q_1} e^{i\alpha})} \tag{5.10}$$

By equations (3.10), (3.3), and (2.6), we obtain a one-soliton solution for the SASDSU(2)GFEs as

$$f_1(\rho, z) = 1/[M_{1H}]_{11}, \qquad g_1(\rho, z) = [M_{1H}]_{12}/[M_{1H}]_{11} \tag{5.11}$$

Similarly, the related matrix elements of the two-soliton solution M_{2H} are

$$[M_{2H}]_{11} = \frac{\mu_1\mu_2}{\rho^2}\varphi - \frac{|\mu_1\mu_2|}{\rho^2 \Delta_H} \left[\frac{|N_H^{(1)}|^2 |m_H^{(2)}|^2}{|\mu_1|^2(|\mu_2|^2 + \rho^2)} - \frac{\overline{N_H^{(1)}} N_H^{(2)} m_H^{(1)} \overline{m_H^{(2)}}}{\mu_1 \overline{\mu_2}(\mu_1 \overline{\mu_2} + \rho^2)} - \frac{\overline{N_H^{(2)}} N_H^{(1)} m_H^{(2)} \overline{m_H^{(1)}}}{\mu_2 \overline{\mu_1}(\mu_2 \overline{\mu_1} + \rho^2)} + \frac{|N_H^{(2)}|^2 |m_H^{(1)}|^2}{|\mu_2|^2(|\mu_1|^2 + \rho^2)} \right]$$

$$[M_{2H}]_{12} = \overline{[M_{2H}]_{21}} = \frac{|\mu_1\mu_2|}{\rho^2}e^{i\alpha} - \frac{|\mu_1\mu_2|}{\rho^2\Delta_H}\left[\frac{\overline{N_H^{(1)}}g_1|m_H^{(2)}|^2}{|\mu_1|^2(|\mu_2|^2+\rho^2)}\right.$$
$$\left. - \frac{\overline{N_H^{(1)}}q_2 m_H^{(1)}\overline{m_H^{(2)}}}{\mu_1\overline{\mu_2}(\mu_1\overline{\mu_2}+\rho^2)} - \frac{\overline{N_H^{(2)}}q_1 m_H^{(2)}\overline{m_H^{(1)}}}{\mu_2\overline{\mu_1}(\mu_2\overline{\mu_1}+\rho^2)} + \frac{\overline{N_H^{2}}q_2|m_H^{(1)}|^2}{|\mu_2|^2(|\mu_1|^2+\rho^2)}\right] \tag{5.12}$$

where

$$\Delta_H = \frac{|m_H^{(1)}|^2|m_H^{(2)}|^2}{(|\mu_1|^2+\rho^2)(|\mu_2|^2+\rho^2)} - \left|\frac{m_H^{(1)}\overline{m_H^{(2)}}}{\mu_1\overline{\mu_2}+\rho^2}\right|^2 = \overline{\Delta_H}$$
$$m_H^{(k)}\overline{m_H^{(l)}} = q_k\overline{q_l}(\varphi - Y_k - \overline{Y_l}) + p_k\overline{q_l}e^{i\alpha} + q_k\overline{p_l}e^{-i\alpha} \tag{5.13}$$
$$N_H^{(k)} = p_k + q_k e^{-i\alpha}(\varphi - Y_k)$$

Thus, by equations (3.10), (3.3), and (2.6), we obtain a two-soliton solution of SASDSU(2)GFEs as

$$f_2(\rho, z) = \rho[M_{2H}]_{11}, \qquad g_2(\rho, z) = V(F_H, G_H) \tag{5.14}$$

where $F_H = 1/[M_{2H}]_{11}$, $G_H = [M_{2H}]_{12}/[M_{2H}]_{11}$.

This process can be continued to obtain a soliton solution of SASD-SU(2)GFEs for each positive integer n, and this soliton solution family and the seed solution (5.8) itself are all new.

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